SIAM REVIEW
Vol. 7, No. 3, July, 1965

SIMULTANEOUS PAIRS OF DUAL INTEGRAL EQUATIONS*

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Introduction. A fruitful method of attack in the solution of mixed boundary value problems is the utilization of integral transform techniques. Satisfaction of the mixed boundary conditions then leads to consideration of one or more pairs of dual integral equations. Oftentimes the solution of these dual integral equations only follows from laborious and complex manipulations. An exception to this is the work of Copson [1] and Lowengrub, Sneddon [2] in which the general solution to the pair of dual integral equations

$$\int_0^\infty \psi(\xi)J_\nu(\xi r)\,d\xi = f_1(r), \qquad 1 < r < \infty, \tag{1a}$$

$$\int_0^\infty \psi(\xi)\xi^{2\alpha}J_\nu(\xi r)\,d\xi = f_2(r), \qquad 0 < r < 1, \tag{1b}$$

is obtained in a simple and straightforward manner.

It is the purpose of this article to demonstrate that the solution techniques in [1], [2] are equally applicable for the simultaneous pairs of dual integral equations

$$\int_0^\infty [a\psi_1(\xi) + \psi_2(\xi)]J_{\nu+2}(\xi r)\,d\xi = f_1(r), \qquad 1 < r < \infty, \tag{2a}$$

$$\int_0^\infty [b\psi_1(\xi) + \psi_2(\xi)]\xi^{2\alpha}J_{\nu+2}(\xi r)\,d\xi = f_2(r), \qquad 0 < r < 1, \tag{2b}$$

$$\int_0^\infty [c\psi_1(\xi) + \psi_2(\xi)]J_\nu(\xi r)\,d\xi = f_3(r), \qquad 1 < r < \infty, \tag{3a}$$

$$\int_0^\infty [\psi_1(\xi) + \psi_2(\xi)]\xi^{2\alpha}J_\nu(\xi r)\,d\xi = f_4(r), \qquad 0 < r < 1. \tag{3b}$$

In (2) and (3), a, b, c are given constants, $\psi_1(\xi)$, $\psi_2(\xi)$ are unknown functions to be determined. It should be noted that for the particular case $a = b = -c = -1$ the simultaneous pairs (2) and (3) may be uncoupled, reducing the problem to consideration of two separate pairs of dual integral equations of form (1). Solution then follows immediately from [1], [2].

It is found convenient to consider (2) and (3) when only one of the nonhomogeneous terms $f_i(r)$, $i = 1, 2, 3, 4$, is nonzero. Since the solution methods depend on whether α is positive or negative, there are then a total of eight cases to be considered. In the following, details are presented for only one case; the results for the other seven are merely stated, since their determination involves nothing novel beyond the example case and manipulations common to [1], [2].

* Received by the editors November 19, 1963, and in revised form August 12, 1964.

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The analysis contained herein is formal, and no attempt is made to supply details of rigor.

Solution. The following lemmas are needed.

LEMMA A. *If* $\lambda > \mu > -1$, *then*

$$\int_0^\infty J_\lambda(a\xi)J_\mu(b\xi)\xi^{1+\mu-\lambda}\,d\xi = \frac{b^\mu(a^2-b^2)^{\lambda-\mu-1}H(a-b)}{2^{\lambda-\mu-1}a^\lambda\Gamma(\lambda-\mu)},$$

where $H(x)$ *is the Heaviside step function.*

This result may be found in Watson [3].

LEMMA B. *Let* $f(x)$, $f'(x)$ *be continuous in* $0 \leqq x \leqq d$. *Let* $0 < \kappa < 1$. *Then the solution of*

$$\int_0^x h(t)(x^2-t^2)^{-\kappa}dt = f(x), \qquad 0 < x < d,$$

is

$$h(t) = \frac{2\sin(\pi\kappa)}{\pi}\frac{d}{dt}\int_0^t xf(x)(t^2-x^2)^{\kappa-1}\,dx.$$

This result is obtained through a transformation of the solution of Abel's integral equation [4].

In the example case considered, α is positive and

$$f_i(r) = 0,\ i = 1, 2, 3;\ f_4(r) \neq 0.$$

Following the lead of Copson [1] the solution form is assumed to be

$$\psi_1(\xi) = \xi^{1-\alpha}\int_0^1 [\phi_1(t)J_{\nu+\alpha}(\xi t) + \phi_2(t)J_{\nu+\alpha+2}(\xi t)]\,dt, \tag{4a}$$

$$\psi_2(\xi) = \xi^{1-\alpha}\int_0^1 [\phi_3(t)J_{\nu+\alpha}(\xi t) + \phi_4(t)J_{\nu+\alpha+2}(\xi t)]\,dt. \tag{4b}$$

Substitution of (4) into (2a) and interchanging the order of integration yields

$$\int_0^1 [a\phi_1(t) + \phi_3(t)]\,dt\int_0^\infty J_{\nu+\alpha}(\xi t)J_{\nu+2}(\xi r)\xi^{1-\alpha}\,d\xi$$
$$+\int_0^1 [a\phi_2(t) + \phi_4(t)]\,dt\int_0^\infty J_{\nu+\alpha+2}(\xi t)J_{\nu+2}(\xi r)\xi^{1-\alpha}\,d\xi = 0, \qquad 1 < r < \infty.$$

Assuming $\nu > -3$ and $\alpha > 0$, Lemma A indicates the second improper integral is zero for $r > 1$. Thereby (2a) is satisfied if

$$\phi_3(t) = -a\phi_1(t). \tag{5}$$

In considering (2b) it is necessary to modify (4) by integrating by parts. Accordingly

$$\psi_1(\xi) = \xi^{1-\alpha} \int_0^1 [\phi_1(t)J_{\nu+\alpha}(\xi t) + \phi_2(t)J_{\nu+\alpha+2}(\xi t)]\, dt$$

$$= \xi^{-\alpha} \int_0^1 \Big\{\phi_1(t)t^{-(\nu+\alpha+1)} \frac{d}{dt} (t^{(\nu+\alpha+1)}J_{\nu+\alpha+1}(\xi t))$$

$$- \phi_2(t)t^{(\nu+\alpha+1)} \frac{d}{dt} (t^{-(\nu+\alpha+1)}J_{\nu+\alpha+1}(\xi t))\Big\}\, dt,$$

and hence

$$(6) \quad \psi_1(\xi) = \xi^{-\alpha}[\phi_1(1) - \phi_2(1)]J_{\nu+\alpha+1}(\xi) - \xi^{-\alpha} \int_0^1 [\Phi_1(t) + \Phi_2(t)]J_{\nu+\alpha+1}(\xi t)\, dt,$$

where it has been assumed that

$$(7) \qquad \lim_{t\to 0+} (\phi_i(t)t^{\nu+\alpha+1}) = 0, \qquad i = 1, 2.$$

In a similar manner

$$\psi_2(\xi) = \xi^{1-\alpha}[\phi_3(1) - \phi_4(1)]J_{\nu+\alpha+1}(\xi) - \xi^{-\alpha} \int_0^1 [\Phi_3(t) + \Phi_4(t)]J_{\nu+\alpha+1}(\xi t)\, dt,$$

and hence

$$(8) \qquad \begin{aligned} \psi_2(\xi) = {} & -\xi^{-\alpha}[a\phi_1(1) + \phi_4(1)]J_{\nu+\alpha+1}(\xi) \\ & - \xi^{-\alpha} \int_0^1 [-a\Phi_1(t) + \Phi_4(t)]J_{\nu+\alpha+1}(\xi t)\, dt, \end{aligned}$$

where it is assumed that

$$\lim_{t\to 0+} (\phi_4(t)t^{\nu+\alpha+1}) = 0.$$

In the preceding the following notation has been adopted:

$$\Phi_{1,3}(t) = t^{\nu+\alpha+1} \frac{d}{dt} (\phi_{1,3}(t)t^{-(\nu+\alpha+1)}); \qquad \Phi_{2,4}(t) = -t^{-(\nu+\alpha+1)} \frac{d}{dt} (\phi_{2,4}(t)t^{\nu+\alpha+1}).$$

Upon substituting (6) and (8) into (2b) it is seen that

$$[(b - a)\phi_1(1) - b\phi_2(1) - \phi_4(1)] \int_0^\infty J_{\nu+\alpha+1}(\xi)J_{\nu+2}(\xi r)\xi^\alpha\, d\xi$$

$$- \int_0^1 [(b - a)\Phi_1(t) + b\Phi_2(t) + \Phi_4(t)]\, dt \int_0^\infty J_{\nu+\alpha+1}(\xi t)J_{\nu+2}(\xi r)\xi^\alpha\, d\xi = 0,$$

$$0 < r < 1.$$

Provided $\alpha < 1$, utilization of Lemma A reduces this to

$$- \int_0^r [(b - a)\Phi_1(t) + b\Phi_2(t) + \Phi_4(t)] \frac{t^{\nu+\alpha+1}2^\alpha}{r^{\nu+2}\Gamma(1 - \alpha)(r^2 - t^2)^\alpha}\, dt = 0,$$

$$0 < r < 1,$$

which is identically satisfied if

$$(b - a)\Phi_1(t) + b\Phi_2(t) + \Phi_4(t) = 0. \tag{9}$$

Consideration of (3a) again necessitates modification of (4). After integrating by parts (4) is expressed in the form

$$\psi_1(\xi) = \xi^{1-\alpha} \int_0^1 \phi_1(t) J_{\nu+\alpha}(\xi t)\, dt - \xi^{-\alpha}\phi_2(1) J_{\nu+\alpha+1}(\xi) - \xi^{-\alpha} \int_0^1 \Phi_2(t) J_{\nu+\alpha+1}(\xi t)\, dt, \tag{10a}$$

$$\psi_2(\xi) = \xi^{1-\alpha} \int_0^1 (-a)\phi_1(t) J_{\nu+\alpha}(\xi t) dt - \xi^{-\alpha}\phi_4(1) J_{\nu+\alpha+1}(\xi) - \xi^{-\alpha} \int_0^1 \Phi_4(t) J_{\nu+\alpha+1}(\xi t)\, dt. \tag{10b}$$

Insertion of (10) into (3a) and interchanging the order of integration leads to

$$\int_0^1 (c - a)\phi_1(t)\, dt \int_0^\infty J_{\nu+\alpha}(\xi t) J_\nu(\xi r) \xi^{1-\alpha}\, d\xi - [c\phi_2(1) + \phi_4(1)] \int_0^\infty J_{\nu+\alpha+1}(\xi) J_\nu(\xi r) \xi^{-\alpha}\, d\xi - \int_0^1 [c\Phi_2(t) + \Phi_4(t)]\, dt \int_0^\infty J_{\nu+\alpha+1}(\xi t) J_\nu(\xi r) \xi^{-\alpha}\, d\xi = 0, \qquad 1 < r < \infty.$$

By Lemma A these three improper integrals are zero and (3a) is satisfied provided $\nu > -1$.

It still remains to satisfy the final relation (3b). Setting

$$\phi_4(t) = -\phi_2(t), \tag{11}$$

it is apparent that

$$\psi_1(\xi) + \psi_2(\xi) = (1 - a)\xi^{1-\alpha} \int_0^1 \phi_1(t) J_{\nu+\alpha}(\xi t)\, dt,$$

and hence

$$\psi_1(\xi) + \psi_2(\xi) = -(1 - a)\xi^{-\alpha}\phi_1(1) J_{\nu+\alpha-1}(\xi) + (1 - a)\xi^{-\alpha} \int_0^1 t^{-(\nu+\alpha+1)} \frac{d}{dt} (\phi_1(t) t^{\nu+\alpha+1}) J_{\nu+\alpha-1}(\xi t)\, dt, \tag{12}$$

assuming

$$\lim_{t\to 0+} (\phi_1(t) t^{\nu+\alpha-1}) = 0.$$

Substitution of (12) in (3b) and interchange of orders of integration yield

$$- (1 - a)\phi_1(1) \int_0^\infty J_{\nu+\alpha-1}(\xi)J_\nu(\xi r)\xi^\alpha d\xi$$

$$+ (1 - a) \int_0^1 t^{-(\nu+\alpha+1)} \frac{d}{dt} (\phi_1(t)t^{\nu+\alpha-1})\, dt \int_0^\infty J_{\nu+\alpha-1}(\xi t)J_\nu(\xi r)\xi^\alpha\, d\xi = f_4(r), \qquad 0 < r < 1.$$

Again from Lemma A this reduces to

$$(1 - a) \int_0^r \frac{\frac{d}{dt} (\phi_1(t)t^{\nu+\alpha-1})2^\alpha}{(r^2 - t^2)^\alpha r^\nu \Gamma(1 - \alpha)}\, dt = f_4(r), \qquad 0 < r < 1.$$

This integral equation is of the type in Lemma B and its solution may be written

$$\frac{d}{dt} (\phi_1(t)t^{\nu+\alpha-1}) = \frac{2^{1-\alpha} \sin (\alpha\pi)\Gamma(1 - \alpha)}{\pi(1 - a)} \frac{d}{dt} \int_0^t \frac{r^{\nu+1}f_4(r)\, dr}{(t^2 - r^2)^{1-\alpha}},$$

where $r^\nu f_4(r)$ and its derivative are assumed continuous. Integrating and using (7) give the solution for $\phi_1(t)$, namely,

$$\phi_1(t) = \frac{2^{1-\alpha}t^{-(\nu+\alpha-1)}}{(1 - a)\Gamma(\alpha)} \int_0^t \frac{r^{\nu+1}f_4(r)\, dr}{(t^2 - r^2)^{1-\alpha}}. \tag{13}$$

The complete solution of this case now follows from the consideration of (7). Employing (11), equation (9) reduces to

$$\frac{d}{dt} (\phi_2(t)t^{\nu+\alpha+1}) = \frac{(b - a)}{(b - 1)} t^{2(\nu+\alpha+1)} \frac{d}{dt} (\phi_1(t)t^{-(\nu+\alpha+1)}),$$

and utilizing (10) leads to the solution for $\phi_2(t)$ in terms of $\phi_1(t)$:

$$\phi_2(t) = \frac{(b - a)}{(b - 1)} \phi_1(t) - \frac{2(b - a)}{(b - 1)} (\nu + \alpha + 1)t^{-(\nu+\alpha+1)} \int_0^t \tau^{\nu+\alpha}\phi_1(\tau)\, d\tau. \tag{14}$$

Equations (5), (11), (13) and (14) in conjunction with (4) represent the final solution of (2) and (3) for the example case.

The solutions for the four cases in which the nonhomogeneous terms $f_2(r)$, $f_4(r)$ are prescribed are next stated. It is assumed that $f_2(r)$, $f_4(r)$ and their derivatives are suitably well-behaved.

Case 1. $f_4(r) \neq 0, \quad f_i(r) = 0, i = 1, 2, 3, \quad \alpha \in (0,1), \quad \nu > -\alpha.$

$$\phi_1(t) = -\frac{\phi_3(t)}{a} = \frac{2^{1-\alpha}t^{-(\nu+\alpha+1)}}{(1 - a)\Gamma(\alpha)} \int_0^t \frac{r^{\nu+1}f_4(r)\, dr}{(t^2 - r^2)^{1-\alpha}},$$

$$\phi_2(t) = -\phi_4(t) = \frac{(b - a)}{(b - 1)} \phi_1(t) - \frac{2(b - a)}{(b - 1)} (\nu + \alpha + 1)t^{-(\nu+\alpha+1)} \int_0^t \tau^{\nu+\alpha}\phi_1(\tau)\, d\tau.$$

Case 2. $f_2(r) \neq 0, \quad f_i(r) = 0, i = 1, 3, 4, \quad \alpha \in (0,1), \quad \nu > -1.$

$$\phi_1(t) = \phi_3(t) = 0,$$

$$\phi_2(t) = -\phi_4(t) = \frac{2^{1-\alpha}t^{-(\nu+\alpha+1)}}{(b-1)\Gamma(\alpha)} \int_0^t \frac{r^{\nu+3}f_2(r)\,dr}{(t^2-r^2)^{1-\alpha}}.$$

Case 3. $f_4(r) \neq 0, \quad f_i(r) = 0, i = 1, 2, 3, \quad \alpha \in (-1,0), \quad \nu > -(1+\alpha).$

$$\phi_1(t) = -\frac{\phi_3(t)}{a} = \frac{2^{-\alpha}t^{-(\alpha+\nu)}}{(1-a)\Gamma(1+\alpha)} \frac{d}{dt} \int_0^t \frac{r^{\nu+1}f_4(r)\,dr}{(t^2-r^2)^{-\alpha}},$$

$$\phi_2(t) = -\phi_4(t) = \frac{(b-a)}{(b-1)}\phi_1(t) - \frac{2(b-a)}{(b-1)}(\nu+\alpha+1)t^{-(\nu+\alpha+1)} \int_0^t \tau^{\nu+\alpha}\phi_1(\tau)\,d\tau.$$

Case 4. $f_2(r) \neq 0, \quad f_i(r) = 0, i = 1, 3, 4, \quad \alpha \in (-1,0), \quad \nu > -1.$

$$\phi_1(t) = \phi_3(t) = 0,$$

$$\phi_2(t) = -\phi_4(t) = \frac{2^{-\alpha}t^{-(\nu+\alpha+2)}}{(b-1)\Gamma(1+\alpha)} \frac{d}{dt} \int_0^t \frac{r^{\nu+3}f_2(r)\,dr}{(t^2-r^2)^{-\alpha}}.$$

When dealing with the four cases in which the prescribed nonhomogeneous terms are $f_1(r)$, $f_3(r)$, it is necessary to adopt a slightly different solution form. Following the example set by Lowengrub, Sneddon [2] this is assumed to be

$$\psi_1(\xi) = \xi^{1-\alpha} \int_1^\infty [\phi_1(t)J_{\nu+\alpha}(\xi t) + \phi_2(t)J_{\nu+\alpha+2}(\xi t)]\,dt, \tag{15a}$$

$$\psi_2(\xi) = \xi^{1-\alpha} \int_1^\infty [\phi_3(t)J_{\nu+\alpha}(\xi t) + \phi_4(t)J_{\nu+\alpha+2}(\xi t)]\,dt. \tag{15b}$$

In conjunction with (15) the solutions for the remaining four cases are as follows.

Case 5. $f_3(r) \neq 0, \quad f_i(r) = 0, i = 1, 2, 4, \quad \alpha \in (0,1), \quad \nu > -\alpha.$

$$\phi_1(t) = -\frac{\phi_3(t)}{a} = \frac{-2^{\alpha}t^{(\nu+\alpha)}}{(c-a)\Gamma(1-\alpha)} \frac{d}{dt} \int_t^\infty \frac{r^{1-\nu}f_3(r)\,dr}{(r^2-t^2)^{\alpha}},$$

$$\phi_2(t) = \phi_4(t) = 0.$$

Case 6. $f_1(r) \neq 0, \quad f_i(r) = 0, i = 2, 3, 4, \quad \alpha \in (0,1), \quad \nu > -\alpha.$

$$\phi_1(t) = -\frac{\phi_3(t)}{a} = \frac{(c-1)}{(c-a)}\phi_2(t) - \frac{(c-1)}{(c-a)}2(\nu+\alpha+1)t^{(\nu+\alpha+1)} \int_t^\infty \phi_2(\tau)\tau^{-(\nu+\alpha+2)}\,d\tau.$$

$$\phi_2(t) = -\phi_4(t) = \frac{2^{\alpha}t^{(\nu+\alpha+2)}}{(1-a)\Gamma(1-\alpha)} \frac{d}{dt} \int_t^\infty \frac{r^{-(\nu+1)}f_1(r)\,dr}{(r^2-t^2)^{\alpha}}.$$

Case 7. $f_3(r) \neq 0, \quad f_i(r) = 0, i = 1, 2, 4, \quad \alpha \in (-1,0), \quad \nu > -\alpha.$

$$\phi_1(t) = -\frac{\phi_3(t)}{a} = \frac{2^{1+\alpha}t^{(\nu+\alpha+1)}}{(c-a)\Gamma(-\alpha)} \int_t^\infty \frac{r^{1-\nu}f_3(r)\,dr}{(r^2-t^2)^{1+\alpha}},$$

$$\phi_2(t) = \phi_4(t) = 0.$$

Case 8. $f_1(r) \neq 0, \quad f_i(r) = 0, i = 2, 3, 4, \quad \alpha \in (-1,0), \quad \nu > -\alpha.$

$$\phi_1(t) = -\frac{\phi_3(t)}{a} = \frac{(c-1)}{(c-a)}\phi_2(t) - \frac{(c-1)}{(c-a)}2(\nu+\alpha+1)t^{(\nu+\alpha+1)}\int_t^\infty \tau^{-(\nu+\alpha+2)}\phi_2(\tau)\,d\tau,$$

$$\phi_2(t) = -\phi_4(t) = \frac{2^{1+\alpha}t^{(\nu+\alpha+3)}}{(a-1)\Gamma(-\alpha)}\int_t^\infty \frac{r^{-(\nu+1)}f_1(r)\,dr}{(r^2-t^2)^{1+\alpha}}.$$

Remarks. Upon noting the relationship

$$\frac{d}{dt}\int_0^t f(u)(t^2-u^2)^\gamma\,du = 2t\gamma\int_0^t \frac{f(u)\,du}{(t^2-u^2)^{1-\gamma}}, \qquad \gamma \in (0,1),$$

and a similar one for

$$\int_t^\infty \frac{f(u)\,du}{(u^2-t^2)^{1-\gamma}},$$

it is seen that the solutions for $\alpha \in (0,1)$ and $\alpha \in (-1,0)$ are identical in form.

For the particular situation $\alpha = 0$, the preceding solution technique is not valid. In fact, when $\alpha = 0$ there is some question as to whether a solution exists. Of course when $a = b = -c = -1$, equations (2) and (3) can be separated into two separate pairs of equations. Since $\alpha = 0$, each pair reduces to a single Fredholm integral equation of the first kind and solution follows immediately upon employing the Hankel inversion theorem.

Examination of the formal solution indicates that solutions do not seem to exist for all values of the constants a, b, c. For certain critical values of these parameters ($a = 1, b = 1, c = a$) the problem is evidently not well posed.

In [5] Erdogan and Bahar have considered the following class of simultaneous dual integral equations:

$$\text{(16a)} \qquad \int_0^\infty \sum_{j=1}^n a_{ij}(x)f_j(x)J_{\mu_i}(xy)\,dx = h_i(y), \qquad 0 < y < 1,$$

$$\text{(16b)} \qquad \int_0^\infty \sum_{j=1}^n b_{ij}(x)f_j(x)J_{\mu_i}(xy)\,dx = g_i(y), \qquad y > 1.$$

Using a generalization of the method developed by Tranter [6] the authors reduced the problem to consideration of an infinite system of linear algebraic equations.

The simultaneous dual integral equations considered herein are a special case of (16). This is readily seen by noting the equivalence between (2) and (16) when $n = 2$:

$$a_{11} = bx^{2\alpha}, \quad a_{12} = a_{21} = a_{22} = x^{2\alpha};$$

$$b_{11} = a, \quad b_{22} = c, \quad b_{12} = b_{21} = 1;$$

$$\mu_1 = \nu + 2, \quad \mu_2 = \nu.$$

In [7] Erdelyi and Sneddon have indicated how the systematic use of fractional integration operators (see [8]) enables a concise statement of solution methods for dual integral equations involving Bessel functions. The solution presented herein certainly could be stated in terms of these operators, thus circumventing many of the computations presented.

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